Name: ____________________

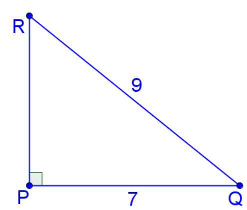
SM3 11.3: Law of Sines

Review: Solving a triangle means finding all of the side and angle measures of the triangle.

Solving right triangles has required the Pythagorean Theorem, $a^2 + b^2 = c^2$, trigonometric functions, $\sin(\,)$, $\cos(\,)$, and $\tan(\,)$, as well as their inverse functions $\arcsin(\,)$, $\arccos(\,)$, and $\arctan(\,)$. Our calculator is typically in "degrees" mode.

Review Example: Given that $m\angle P = 90°, r = 7, p = 9$, solve ΔPQR.

Step 1) Sketch a reasonable representation of the triangle to help decide which pieces of information played each role in the triangle:



Step 2) If you're given two sides, find the third side with the Pythagorean Theorem. If you're given two angles, find the third angle by subtraction.

We're given two sides, so it's Pythagorean Theorem time!

$$a^2 + b^2 = c^2$$
$$q^2 + 7^2 = 9^2$$
$$q^2 + 49 = 81$$
$$q^2 = 32$$
$$q \approx 5.7$$

Step 3) If a side is still missing, use a trig evaluation. If an angle is still missing, use an inverse trig evaluation.

We're still missing angles, so we'll need an inverse trig evaluation!

$$\cos Q = \frac{7}{9}$$
$$m\angle Q = \arccos\left(\frac{7}{9}\right)$$
$$m\angle Q \approx 38.9°$$

Step 4) If a side is still missing, use another trig evaluation. If an angle is still missing, use subtraction.

We're missing an angle, so we'll need to subtract!

$$m\angle P + m\angle Q + m\angle R = 180°$$
$$90° + 38.9° + m\angle R \approx 180°$$
$$m\angle R \approx 51.1°$$

Step 5) Write your solution as the set of values that you solved for.

$$\boxed{q \approx 5.7, m\angle Q \approx 38.9°, m\angle R \approx 51.1°}$$

Note: Impossible cases to solve exist (e.g., the hypotenuse is shorter than a leg, or sum of the legs is smaller than the hypotenuse, etc.).

The time has come to open up your ability to solve any triangle, not just right triangles. In order to solve a triangle, you'll need to find 3 measurements of any combination of sides and angles.

The solution of a triangle (Latin: *solutio triangulorum*) is the historical term for solving the main trigonometric problem of finding the characteristics of a triangle (angles and lengths of sides), when some of these are known. Depending on the measurements you have, you can determine how many solutions you will have, if you even have any at all.

What happens if we know the value of …

… all 3 sides (SSS)

When the sides' lengths are fixed, the angles that hold the sides in position are also fixed. As a result, there will be only 1 solution to the triangle. Begin with the Law of Cosines.

… all 3 angles (AAA)

While the angle measures are fixed, because the triangle could be dilated to a different size, we'll have infinitely many solutions to the triangle. We don't expect you to find these solutions.

… two angles and one side (AAS or ASA)

Because the angles of a triangle always sum to $180°$ and we know two of the angles, we also know the third angle using subtraction. Since we have the length of one side, our triangle is locked into a certain size. This gives us only 1 solution. Begin with the Law of Sines.

… two sides and the angle between them (SAS)

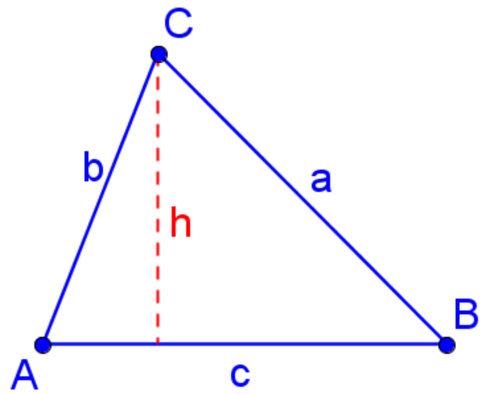
The angle aims the two sides, and they have fixed lengths. There will only be one way to connect the last side of the triangle. Hence, there will be only 1 solution. Begin with the Law of Cosines.

…two sides and an angle that is not between them (SSA)

Depending on the values, a variety of scenarios can take place. We'll need to explore this case together and discover how to proceed.

Law of Sines

Given ΔABC, it is possible to construct the height of the triangle from C to c. Call this h.



$\sin A = \frac{h}{b}$	Definition of $\sin(\,)$.
$h = b \sin A$	Multiplication
$\sin B = \frac{h}{a}$	Definition of $\sin(\,)$.
$h = a \sin B$	Multiplication
$b \sin A = a \sin B$	Substitution
$\frac{\sin A}{a} = \frac{\sin B}{b}$	Division

By constructing a height from B to b, we could similarly prove that $\frac{\sin A}{a} = \frac{\sin C}{c}$

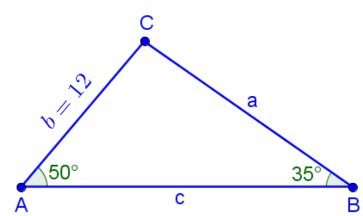
By constructing a height from A to a, we could similarly prove that $\frac{\sin B}{b} = \frac{\sin C}{c}$

Law of Sines: Given ΔABC, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Example: Solve ΔABC given that $\text{m}\angle A = 50°$, $\text{m}\angle B = 35°$, and $b = 12$.

Step 1) Sketch the triangle and determine which type of information was given.

It appears we've been given AAS information. We'll proceed with use of the Law of Sines to solve.



Step 2) We only use two ratios of the Law of Sines at one time. Since we know $\text{m}\angle B$ and b, we'll use the ratio that contains both of those terms. The other known information we have is $\text{m}\angle A$, so we'll use the ratio that contains it as well.

$\frac{\sin A}{a} = \frac{\sin B}{b}$ Law of Sines

$\frac{\sin 50°}{a} = \frac{\sin 35°}{12}$ Substitution

$a = \frac{12 \sin 50°}{\sin 35°}$ Solve for a

$a \approx 16.0$ Use a calculator, round to nearest tenth.

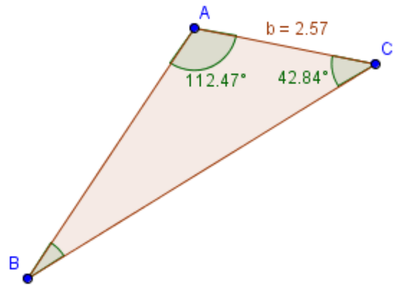
Step 3) We can find the missing angle by subtraction, and then use the Law of Sines again to find the last missing length. $\text{m}\angle C = 95°$, $c \approx 20.8$

$a \approx 16.0$, $\text{m}\angle C = 95°$, and $c \approx 20.8$

Problems: Find the missing measurements to the nearest hundredth using the Law of Sines:

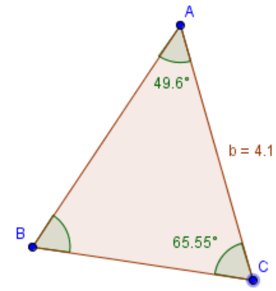
1)

$m\angle B =$
$a =$
$c =$



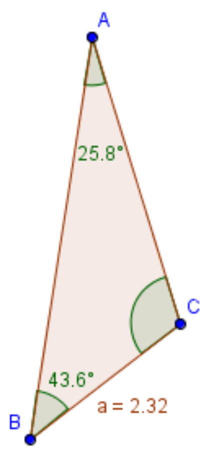
2)

$m\angle B =$
$a =$
$c =$



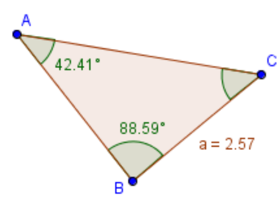
3)

$m\angle C =$
$b =$
$c =$

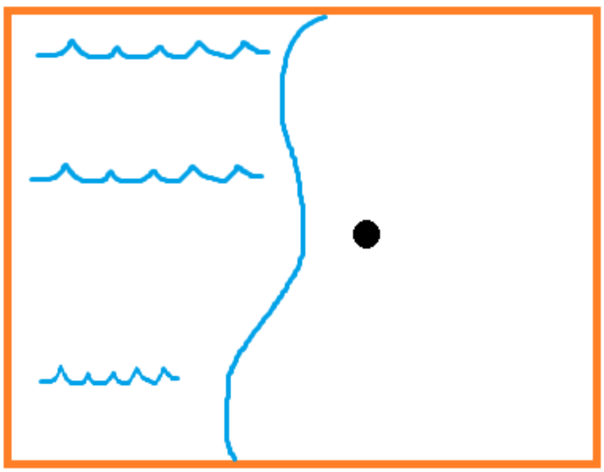


4)

$m\angle C =$
$b =$
$c =$



5) Josie, Mckenna, and Whitney take a trip to the California coastline during the summer to enjoy some time at the beach (and to work on their summer calculus homework in a more pleasant environment). Whitney wants to go for a swim, Josie fancies a nap on the beach, and Mckenna decides to study limit notation in their hotel room. Josie and Whitney walk down to the shore and Josie finds a suitable spot to doze off. Whitney runs due northwest from Josie, splashing into the water. As Whitney gets about 50 feet from Josie, Josie notices a rather large fin the in water, due west! Josie screams for Whitney to look out and points toward the fin, and Whitney looks back to Josie then turns 110 degrees clockwise and spots the fin. Whitney is frozen in fear; the perceived shark pauses, anticipating its next move.



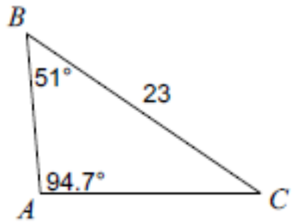
- Draw a point in the water that represents Whitney's location.
- Connect Josie's point and Whitney's point with a line segment.
- Draw a fin in the water.
- Connect the fin to both points with line segments.
- Appropriately label the vertices and sides of the triangle.
- Add known information to the picture.

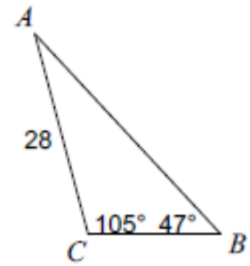
Use the Law of Sines to determine how far apart the fin and Whitney are.

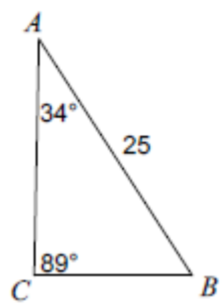
6) Hearing a scream, Mckenna walks onto the patio outside of her well-built hotel room on the 8^{th} floor (approximately 80 feet above the ground). Mckenna sees the fin in the water near her classmate. The angle of depression she can view the fin with is 50 degrees. Mckenna finds a new solution to the question "when will I ever use this?" by summoning superhuman strength and hurling her calculus book from the patio, over the beach, at the base of the fin (assume the textbook travels in a straight line)!

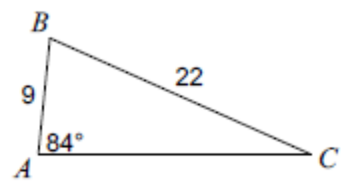
- Sketch the triangular relationship between Mckenna's position, the fin's position, and the base of the hotel.
- Label the points and sides of the triangle. Add known information to the picture.
- Use the Law of Sines to determine how far Mckenna threw the textbook.

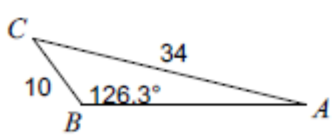
Solve each triangle. Round your answers to the nearest tenth.

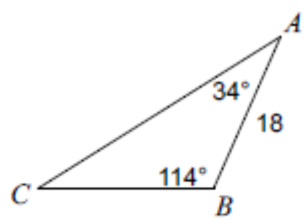
7) 

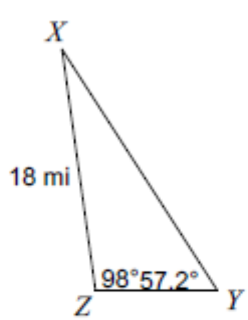
8) 

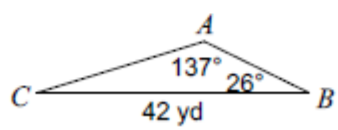
9) 

10) 

11) 

12) 

13) 

14) 

15) $m\angle B = 91°, m\angle C = 59°, b = 28$

16) $m\angle C = 91°, m\angle B = 74°, a = 7$

17) $m\angle A = 88°, c = 30, a = 34.4$

18) $m\angle C = 28°, m\angle A = 7.7°, c = 28.2$

19) $m\angle C = 58°, m\angle A = 89.2°, b = 13$

20) $m\angle B = 123°, m\angle A = 23°, c = 10$

21) In $\triangle KHP$, $m\angle K = 22°, m\angle H = 114.4°, p = 25$ cm

22) In $\triangle STR$, $m\angle S = 100°, m\angle T = 32°, s = 46.4$ km